

Scaling and front dynamics in Ising quantum chains

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Abstract. We study the relaxation behaviour of the quantum Ising chain, focusing our attention onto the non-equilibrium dynamics of the transverse magnetization. The initial states, from which the magnetization relaxes, are product states such as those generated by setting in contact several systems, each initially equilibrated at a given temperature. Due to the free fermionic structure of the chain, the dynamics of the transverse magnetization is easily expressed in a compact form. In the completely factorized initial state, corresponding to a situation where all the spins are thermalized independently, we obtain in the scaling limit the Green function associated to the transverse magnetization. The relaxation behaviour is also considered in the system-bath case, where part of the chain called the system is thermalized at a temperature T_s and the remaining part is at a temperature T_b . The magnetization profiles show a scaling behaviour. Moreover, in the extreme case $T_b = \infty$ and $T_s = 0$, it is shown that the magnetization relaxes in quantized steps in the strong transverse field region.

PACS. 75.40.Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.) – 05.70.Ln Nonequilibrium and irreversible thermodynamics

1 Introduction

Out of equilibrium properties of quantum chains have been the purpose of several recent studies (see Ref. [1] for a recent review). In particular, after precursor studies [2–4] at the end of the sixties, some renewed interest has been put on the relaxation properties of free fermionic quantum chains initially prepared in nonequilibrium states [5–9, 12, 13]. The influence of randomness was also considered in this context [14]. In this contribution, we present the relaxation of the transverse magnetization of an Ising quantum chain. The system is initially prepared in a factorized state $\rho(0) = \prod_j \rho_j$ where the density matrices $\rho_j \propto e^{-\beta_j \mathcal{H}_j}$ are canonical states of parts of the chain thermalized each at a given temperature β_j^{-1} . One can look at that state as obtained from a chain initially split into several non-interacting terms: $\mathcal{H}_0 = \sum_j \mathcal{H}_j$, each in contact with a specific heat bath. At time $t = 0$, the interactions between the different parts are switched on, the heat baths removed, so that the time evolution of the system is generated quantum mechanically by the total Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \sum_j \mathcal{H}_{j,j+1}^I$, such that the system will relax from the initial factorized state $\rho(0)$ toward a new state. In particular, we will focus our attention on the

behaviour of the initial interface by following the front dynamics generated by the initial thermal inhomogeneity.

In this paper, we present the transverse magnetization profiles obtained on two different situations. The first considered is a situation where the chain is split into two parts, one at low temperature called the “system”, the other at high temperature called the “bath”. So, in the system-bath case, we concentrate on the front dynamics generated by the interface between the system and the bath parts [6, 12, 13]. The other situation we have considered in this study is the droplet-like case, where a low temperature finite part of size N is put in contact at both ends with high temperature infinite chains. In this case, we concentrate on the large time $t \gg N$ relaxation of the initial non-equilibrium transverse magnetization, which shows a scaling behaviour.

The paper is organized as follows: in the next section we present the dynamics of the Ising quantum chain solved in terms of Clifford operators [12]. One may notice here that the choice of Clifford operators is somehow arbitrary, in the sense that it is not necessary in order to solve the dynamics. It is more common that people use Fermion algebra instead of Clifford algebra. We show how we calculate the expectation values of several physical quantities, paying a special attention to the transverse magnetization. Section 5 contains our results and we end with a short summary and discussion of them.

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2 Dynamics

The Hamiltonian of the system under consideration is given by

$$\mathcal{H} = -\frac{1}{2} \sum_{k=1}^{L-1} \sigma_k^x \sigma_{k+1}^x - \frac{h}{2} \sum_{k=1}^L \sigma_k^z \quad (1)$$

where the σ s are the Pauli matrices and where h is the transverse field pointing in the z -direction. As it is well known, this Hamiltonian can be fermionized thanks to the Jordan-Wigner transformation [15]:

$$\sigma_n^x = \prod_{j=1}^{n-1} (-\sigma_j^z) \Gamma_n^1 \quad (2)$$

$$\sigma_n^y = -\prod_{j=1}^{n-1} (-\sigma_j^z) \Gamma_n^2 \quad (3)$$

so that

$$\sigma_n^z = -i \Gamma_n^2 \Gamma_n^1 \quad (4)$$

where the Γ s are Clifford operators satisfying the anti-commutation rules

$$\{\Gamma_n^i, \Gamma_m^j\} = 2\delta_{n,m}\delta_{i,j}. \quad (5)$$

The Hamiltonian can then be diagonalized by a linear canonical transformation of the Clifford operators [16,12] and the dynamics of the lattice operators is easily solved, see the appendix for more details, and leads to

$$\Gamma_n^j(t) = \sum_{k,i} \langle \Gamma_k^i | \Gamma_n^j(t) \rangle \Gamma_k^i \quad (6)$$

where the time-dependent contractions $\langle \Gamma_k^i | \Gamma_n^j(t) \rangle$ are explicitly given in the appendix. They depend on the eigenvectors and eigenvalues of the canonical transformation. Since all the operators of the theory can be expressed in terms the operators Γ s, their dynamics is formally known.

3 Expectation values

In this work, we have considered systems prepared in factorized initial states of the form

$$\rho(0) = \prod_j \rho_j \quad (7)$$

where the initial density matrix $\rho(0)$ is given by a product of smaller density matrices ρ_j , corresponding to initially non-interacting subsystems. To be more specific, we have used the canonical initial distributions

$$\rho_j = \frac{1}{Z_j} \exp(-\beta_j \mathcal{H}_j) \quad (8)$$

where β_j is the inverse temperature of the subsystem \mathcal{H}_j which is a part of the full Hamiltonian \mathcal{H} . Z_j is the corresponding partition function. The expectation value of an observable \mathcal{O} at time t is given by

$$\langle \mathcal{O} \rangle(t) = \text{Tr} \left\{ \mathcal{O}(t) \prod_j \rho_j \right\} \quad (9)$$

where $\mathcal{O}(t)$ is the operator associated to the observable \mathcal{O} in the Heisenberg picture.

In particular, in the following we will pay some attention on the two-temperatures problem

$$\rho(0) = \rho_s \rho_b \quad (10)$$

with a small system of size N initially prepared at a temperature T_s putted in contact at one end with a big system of size M (called the bath) at a temperature T_b . After the contact, the time evolution will be governed by the Hamiltonian (1) with $L = N + M$. The expectation value takes the form

$$\langle \mathcal{O} \rangle(t) = \frac{1}{Z_s Z_b} \text{Tr} \{ \mathcal{O}(t) e^{-\beta_s \mathcal{H}_s} e^{-\beta_b \mathcal{H}_b} \} \quad (11)$$

with $\mathcal{O}(t) = e^{i\mathcal{H}t} \mathcal{O} e^{-i\mathcal{H}t}$ and where the total Hamiltonian is split into

$$\mathcal{H} = \mathcal{H}_s + \mathcal{H}_b + \mathcal{H}^I \quad (12)$$

with

$$\mathcal{H}_s = -\frac{1}{2} \sum_{k=1}^{N-1} \sigma_k^x \sigma_{k+1}^x - \frac{h}{2} \sum_{k=1}^N \sigma_k^z, \quad (13)$$

$$\mathcal{H}_b = -\frac{1}{2} \sum_{k=N+1}^{N+M-1} \sigma_k^x \sigma_{k+1}^x - \frac{h}{2} \sum_{k=N+1}^{N+M} \sigma_k^z, \quad (14)$$

and where $\mathcal{H}^I = -\frac{1}{2} \sigma_N^x \sigma_{N+1}^x$ is the initial interface interaction term. For example, in the situation where the system consists of only one spin, expressing the bath part in terms of its diagonal fermionic operators η^\dagger, η , one has for the total Hamiltonian

$$\mathcal{H} = -\frac{h}{2} \sigma_1^z + \sum_{q=1}^M \varepsilon_q^b \eta_q^\dagger \eta_q - \frac{1}{2} \sigma_1^x \sum_{q=1}^M \phi_q^b(1) (\eta_q^\dagger + \eta_q), \quad (15)$$

where $\phi_q(1)$ refers for the first bath site. The total Hamiltonian thus obtained is the fermionic bath version of the so-called ‘‘spin-boson’’ model [10,11]. The bath degrees of freedom act on the two-state system by the linear coupling $-\frac{1}{2} \sigma_1^x \xi = -\frac{1}{2} \sigma_1^x \sum_{q=1}^M \phi_q^b(1) (\eta_q^\dagger + \eta_q)$. The present study can be interpreted in this context.

To evaluate the expectation value $\langle \mathcal{O} \rangle(t)$, one first has to express the operator $\mathcal{O}(t)$ in terms of the elementary time-independent Γ_n^ν Clifford operators. Then, one will have to evaluate expressions of the form

$$\frac{1}{Z_s Z_b} \text{Tr} \{ \Gamma_{k_1}^{i_1} \dots \Gamma_{k_n}^{i_n} e^{-\beta_s \mathcal{H}_s} e^{-\beta_b \mathcal{H}_b} \} \quad (16)$$

which can be calculated explicitly noticing that the Hamiltonians \mathcal{H}_s and \mathcal{H}_b can be expressed in terms of the lattice operators Γ_k .

4 Transverse magnetization

4.1 System plus bath initial state

The transverse magnetization is very simply expressed in terms of the Clifford operators, since it is a local quantity.

One has in the Heisenberg picture

$$\sigma_n^z(t) = -i\Gamma_n^2(t)\Gamma_n^1(t). \quad (17)$$

The expectation value of the transverse magnetization, with initial state $\rho(0) = \rho_s\rho_b$, where the first N spins of the chain are prepared in the canonical state ρ_s while the M remaining spins are in ρ_b , is

$$\langle \sigma_n^z \rangle(t) = -i \sum_{\nu, \nu'=1}^2 \sum_{k, k'=1}^{N+M} \langle \Gamma_k^\nu | \Gamma_n^2(t) \rangle \langle \Gamma_{k'}^{\nu'} | \Gamma_n^1(t) \rangle \times \text{Tr}\{\rho_s\rho_b\Gamma_k^\nu\Gamma_{k'}^{\nu'}\}. \quad (18)$$

The time contractions $\langle \Gamma_k^\nu | \Gamma_n^\mu(t) \rangle$ are known and given in terms of the ϕ, ψ which are obtained by the diagonalization of the T matrix. What remains to be done is to evaluate the trace $\text{Tr}\{\rho_s\rho_b\Gamma_k^\nu\Gamma_{k'}^{\nu'}\}$. One distinguishes three different situations: Γ_k^ν and $\Gamma_{k'}^{\nu'}$ are in the system part, Γ_k^ν and $\Gamma_{k'}^{\nu'}$ are in the bath part, and finally one is in the system while the other is in the bath. In the first case, noticing that

$$\Gamma_k^{1,2} = \Gamma_{s,k}^{1,2} \quad \forall k = 1 \dots N \quad (19)$$

where the $\Gamma_{s,k}$ s are the Clifford operators associated to the first part of the chain (disconnected from the rest) of size N , one has for the trace term

$$\text{Tr}\{\rho_s\rho_b\Gamma_k^\nu\Gamma_{k'}^{\nu'}\} = \text{Tr}_s\{\rho_s\Gamma_{s,k}^\nu\Gamma_{s,k'}^{\nu'}\} \quad (20)$$

The same considerations, together with the fact that

$$\Gamma_{b,k}^{1,2} = Q_s\Gamma_{k+N}^{1,2} \quad (21)$$

where the $\Gamma_{b,k}$ s are associated to the bath part and where $Q_s = \prod_{j=1}^N(-\sigma_j^z)$ is the charge operator of the system part, leads to the expression

$$\text{Tr}\{\rho_s\rho_b\Gamma_{k+N}^\nu\Gamma_{k'+N}^{\nu'}\} = \text{Tr}_b\{\rho_b\Gamma_{b,k}^\nu\Gamma_{b,k'}^{\nu'}\} \quad (22)$$

with the indices k and k' running from 1 to M . Finally, in the last case with one Clifford operator in each sector, it is easy to show that the trace is vanishing since it is proportional to terms of the form $\text{Tr}_b\{e^{-\beta b i \frac{\epsilon_{b,q}}{2} \gamma_{b,q}^1 \gamma_{b,q}^2} \gamma_{b,q}^\nu\} = 0$. Taking into account all the various contributions, one obtains for the expectation value of the transverse magnetization

$$\langle \sigma_n^z \rangle(t) = S_n(t) + B_n(t) \quad (23)$$

where

$$S_n(t) = \sum_{k, k'=1}^N [\langle \Gamma_k^1 | \Gamma_n^1(t) \rangle \langle \Gamma_{k'}^2 | \Gamma_n^2(t) \rangle - \langle \Gamma_k^1 | \Gamma_n^2(t) \rangle \langle \Gamma_{k'}^2 | \Gamma_n^1(t) \rangle] (\mathbf{S})_{k, k'} \quad (24)$$

and

$$B_n(t) = \sum_{k, k'=1}^M [\langle \Gamma_{k+N}^1 | \Gamma_n^1(t) \rangle \langle \Gamma_{k'+N}^2 | \Gamma_n^2(t) \rangle - \langle \Gamma_{k+N}^1 | \Gamma_n^2(t) \rangle \langle \Gamma_{k'+N}^2 | \Gamma_n^1(t) \rangle] (\mathbf{B})_{k, k'} \quad (25)$$

with the canonical contractions of the system

$$(\mathbf{S})_{k, k'} = - \sum_{q=1}^N \phi_{s,q}(k) \psi_{s,q}(k') \tanh\left(\beta_s \frac{\epsilon_{s,q}}{2}\right) \quad (26)$$

and the canonical contractions of the bath

$$(\mathbf{B})_{k, k'} = - \sum_{q=1}^M \phi_{b,q}(k) \psi_{b,q}(k') \tanh\left(\beta_b \frac{\epsilon_{b,q}}{2}\right). \quad (27)$$

4.2 General expressions

The matrices \mathbf{S} and \mathbf{B} contain the initial state properties. They can be rewritten more explicitly in terms of the spin operators, utilizing equations (2) and (3), as

$$(\mathbf{S})_{k, k'} = \text{Tr}\{\rho_s i \Gamma_{s,k}^1 \Gamma_{s,k'}^2\} = -\text{Tr}\{\rho_s i \sigma_k^x Q_k^s Q_{k'}^s \sigma_{k'}^y\} \quad (28)$$

and

$$(\mathbf{B})_{k, k'} = \text{Tr}\{\rho_b i \Gamma_{b,k}^1 \Gamma_{b,k'}^2\} = -\text{Tr}\{\rho_b i \sigma_{N+k}^x Q_k^b Q_{N+k'}^b \sigma_{N+k'}^y\} \quad (29)$$

with $Q_k^s \equiv \prod_{j=1}^{k-1}(-\sigma_j^z)$ and $Q_k^b \equiv \prod_{j=1}^{k-1}(-\sigma_{N+j}^z)$. Defining the $(N+M) \times (N+M)$ matrix \mathbf{I} as

$$\mathbf{I} = \begin{pmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{B} \end{pmatrix} \quad (30)$$

and the time-dependent element

$$P_{n, n'}^{k, k'}(t) = \langle \Gamma_k^1 | \Gamma_n^1(t) \rangle \langle \Gamma_{k'}^2 | \Gamma_{n'}^2(t) \rangle - \langle \Gamma_k^1 | \Gamma_{n'}^2(t) \rangle \langle \Gamma_{k'}^2 | \Gamma_n^1(t) \rangle, \quad (31)$$

it is possible to write the expectation value of the transverse magnetization as

$$\langle \sigma_n^z \rangle(t) = \sum_{k, k'=1}^{L=N+M} P_{n, n'}^{k, k'}(t) (\mathbf{I})_{k, k'}. \quad (32)$$

It is easy to generalize that expression for initial product canonical states of the form $\prod_{j=1}^K \rho_j$ since in this case, the initial matrix \mathbf{I} takes the form

$$\mathbf{I} = \begin{pmatrix} \mathbf{I}_1 & 0 & \dots & 0 \\ 0 & \mathbf{I}_2 & & \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{I}_K \end{pmatrix} \quad (33)$$

where \mathbf{I}_j is the initial matrix of the j -subsystem. The transverse magnetization is given by (32).

On the same lines, we have

$$\langle \sigma_n^x \sigma_{n+1}^x \rangle(t) = \sum_{k, k'=1}^L P_{n+1, n}^{k, k'}(t) (\mathbf{I})_{k, k'}. \quad (34)$$

and

$$\langle \sigma_n^y \sigma_{n+1}^y \rangle(t) = \sum_{k, k'=1}^L P_{n, n+1}^{k, k'}(t) (\mathbf{I})_{k, k'} \quad (35)$$

for the bond correlations.

5 Transverse magnetization profiles

5.1 Completely factorized initial state

In the completely factorized initial state

$$\rho(0) = \rho_j^{\otimes L} \quad (36)$$

where all the spins are thermalized independently with local Hamiltonian $\mathcal{H}_n = -\frac{h}{2}\sigma_n^z$, the initial matrix \mathbf{I} is diagonal:

$$\mathbf{I}_{k,k'} = \delta_{k,k'} \langle \sigma_k^z \rangle(0) \quad (37)$$

where $\langle \sigma_k^z \rangle(0)$ is the initial transverse magnetization. In this case, the transverse magnetization at later time is given by

$$\langle \sigma_n^z \rangle(t) = \sum_k P_{n,n}^{k,k}(t) \langle \sigma_k^z \rangle(0). \quad (38)$$

In the thermodynamical limit $L \rightarrow \infty$, the contractions $\langle \Gamma_k^\nu | \Gamma_n^\mu(t) \rangle$ depend only on the difference $n-k$ so that the transverse magnetization is given by a discrete convolution product

$$\langle \sigma_n^z \rangle(t) = \sum_k F_t(n-k) \langle \sigma_k^z \rangle(0). \quad (39)$$

where the Green function $F_t(n-k) = P_{n,n}^{k,k}$. At the critical point, $h = 1$, the basic contractions are expressed in terms of Bessel functions and one can show that [1,12]

$$F_t(p) = J_{2p}^2(2t) - J_{2p+1}(2t)J_{2p-1}(2t). \quad (40)$$

In the continuum limit, one can show that the Green function $F_t(p) = \frac{1}{t} f\left(\frac{p}{t}\right)$ with the scaling function [12]

$$f(u) = \frac{1}{\pi} \sqrt{1-u^2} \quad |u| \leq 1 \quad (41)$$

and zero otherwise. Depending on the initial state, one will have different profiles for the transverse magnetization which can be in principle explicitly obtained by performing the above convolution product. It is interesting to notice that the time behaviour of the transverse magnetization in this case is basically the same as in the situation when the starting initial state is a pure state of the form

$$|\Psi\rangle = |\dots \sigma_k \sigma_{k+1} \dots\rangle \quad (42)$$

where σ_k is the value of the z -component spin at site k [9,12]. One can see references [1,12] for more details and physical examples.

5.2 Two-temperature state

We turn now to the more complex case where part of the chain is thermalized at inverse temperature β_s and the rest at inverse temperature β_b . The transverse magnetization is given by the general formula (32) with an initial value matrix \mathbf{I} which has non-diagonal terms, taking into account that within the system part, the spins are interacting with each other and consequently have non-vanishing

correlations, the same being true for the bath-part of the chain.

In the following, we consider two different situations. The first is concerned with the interface behaviour, that is mainly how the jump of the transverse magnetization at the boundary between the system and the bath parts will spread out. The second situation deals with the case of a finite system, of N interacting spins, initially at low temperature T_s in contact at both ends with infinite chains at high temperature T_b . In this case, the initially large transverse magnetization of the system (see the Hamiltonian (1)) will decay toward the small bath value as time is evolving.

Consider the first situation. In order to avoid boundary terms, we will take the thermodynamical limit where, both system and bath sizes are sent to infinity. The expectation value of the transverse magnetization is given by

$$\langle \sigma_n^z \rangle(t) = \sum_{i,k} F_t^i(n-k) \mathbf{I}_{k,k+i} \quad (43)$$

where at the critical field value $h = 1$, the Green functions are expressed in terms of Bessel functions:

$$F_t^i(p) = (-1)^i [J_{2p}(2t)J_{2(p-i)}(2t) - J_{2p+1}(2t)J_{2(p-i)-1}(2t)]. \quad (44)$$

The relation giving the transverse magnetization decomposes into a sum of discrete convolution products

$$\langle \sigma_n^z \rangle(t) = \sum_i (F_t^i \star \mathbf{I}^i)(n) \quad (45)$$

with Green functions F_t^i defined above. In the extreme case where $T_s = 0$ and $T_b = \infty$, we arrive at a further simplification since in the initial \mathbf{I} matrix (30), the \mathbf{B} matrix is vanishing and the elements of the system matrix \mathbf{S} are simply $(\mathbf{S})_{ij} = -\sum_{q=1}^{\infty} \phi_{s,q}(i)\psi_{s,q}(j)$. In Figure 1 we show the rescaled transverse magnetization profile obtained numerically by exact diagonalization of the Hamiltonians \mathcal{H}_s , \mathcal{H}_b and \mathcal{H} .

We have the very simple expression:

$$\langle \sigma_n^z \rangle(t) = m_{bulk}^{z,sys} \left[\frac{1}{2} + f\left(\frac{n}{t}\right) \right] \quad (46)$$

where $m_{bulk}^{z,sys} = 2/\pi$ is the initial bulk magnetization of the system and where the scaling function $f(u)$ is given by

$$f(u) = \begin{cases} \frac{1}{2} & u \leq -1 \\ -\frac{1}{2}u & -1 < u < 1 \\ -\frac{1}{2} & u \geq 1. \end{cases} \quad (47)$$

One may notice that since the excitations travel with velocity $c(h=1) = 1$, one expects effects only in the causal region $-1 < \ell/t < 1$ where ℓ is the distance from the initial interface. Outside the causal region, both system and bath parts are still equilibrated and nothing is changed

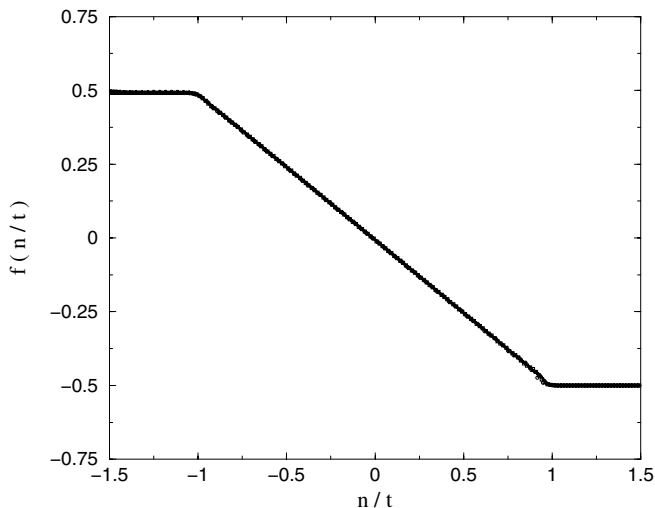


Fig. 1. Scaling function of the transverse magnetization obtained for times $t = 30, 60, 90, 120$ on chains of total size $L = 200$.

since they are at equilibrium. The remarkable linearity of the profile is lost as soon as we have a departure from infinite and vanishing temperature. For finite temperature cases, the profile is rounded.

In the off-critical region, $h > 1$, the transverse magnetization profile, in the case $T_s = 0$ and $T_b = \infty$, shows a feature that was already remarked in the XX-chain [17] context. Indeed, as one can see in Figure 2 the magnetization relaxes in quantized steps. That is, if one concentrates near the front entering into the bath part (which has a vanishing magnetization, since it is at infinite temperature), one finds a staircase like structure with constant area steps. If one defines

$$m(\ell, t) \equiv \langle \sigma_\ell^z \rangle(t) - \langle \sigma_{bath}^z \rangle, \quad (48)$$

as it is clearly shown in Figure 2, the magnetization deviation from the bath magnetization $\langle \sigma_{bath}^z \rangle = 0$ has a scaling behaviour of the form

$$m(\ell, t) = t^{-1/3} g\left(\frac{\ell - t}{t^{1/3}}\right) \quad (49)$$

where again ℓ measures the distance from the initial interface, that is the interface between the system and the bath. Numerically, as seen from Figure 2, it seems that the envelope of the scaling function g is a simple square-root, so that we have the conjecture

$$m(\ell, t) = At^{-1/3} \sqrt{\frac{t - \ell}{t^{1/3}}} \quad \ell < t, \quad (50)$$

where A is a function of the field strength h .

The scaling form (49) implies that since the width of the steps increases as $t^{1/3}$ and their height decreases as $t^{-1/3}$, the area of the steps is indeed constant during time. Each step carries a definite magnetic moment $\mu(h)$ depending on the field strength h . As h is increased toward

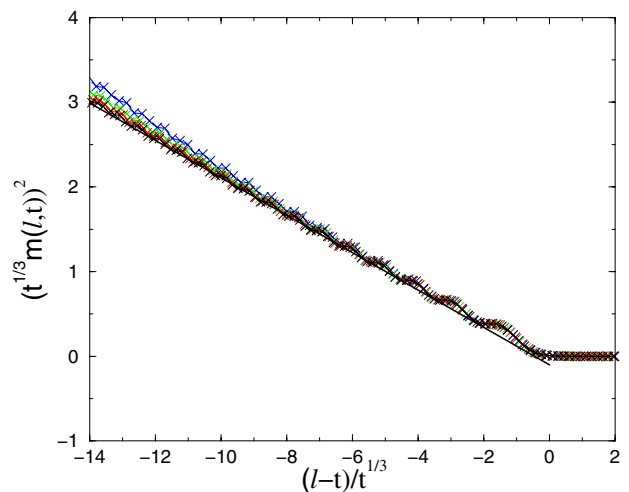


Fig. 2. Scaling of the transverse magnetization deviation $m(\ell, t)$ obtained for times $t = 80, 120, 160, 200$ at transverse field $h = 10$.

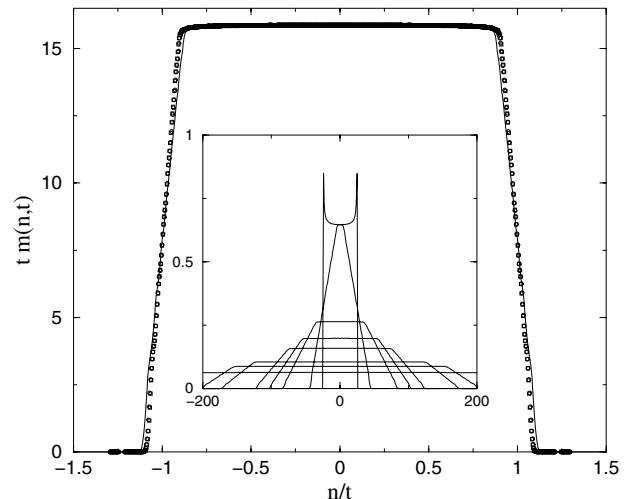


Fig. 3. Rescaled transverse magnetization with $T_s = 0$, $T_b = \infty$ and at $h = 1$ for different times t . Inset: Transverse magnetization profiles at different times.

infinity, the magnetic moment carried by each step reaches the value one (found numerically), which is reminiscent of the fact that in this case, an up spin from the system initial state $|\Psi\rangle = |\uparrow\uparrow \dots \uparrow\rangle$ (see the Hamiltonian (1)) is injected into the bath and carried by one step.

This simple particle-like picture does not apply at the critical field value $h = 1$, since in this case, the initial system state is a critical one with long range correlations. We have seen numerically that the quantized steps picture is not present in the case $h < 1$.

Let us turn now to the finite system of size N , initially at vanishing temperature, in contact at both ends with infinite chains at infinite temperature. We will refer to this situation as the droplet situation. In Figure 3, we show the rescaled transverse magnetization profile obtained numerically at the critical field value $h = 1$. We see two different

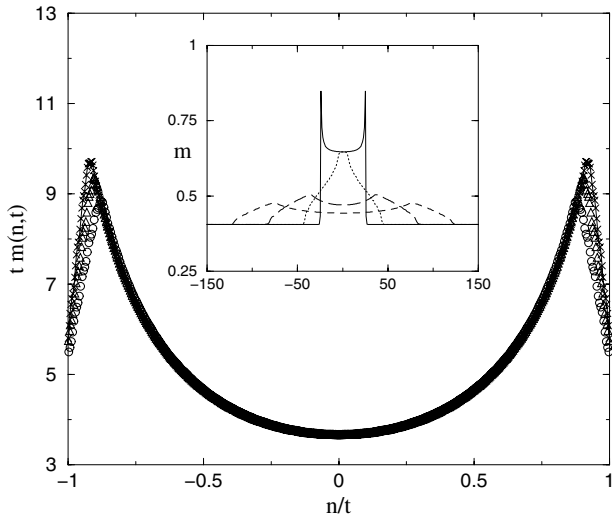


Fig. 4. Rescaled transverse magnetization with $T_s = 0$, $T_b = 1$ and at $h = 1$ for different times t . Inset: Transverse magnetization profiles at different times.

regimes. In the first stage, for times smaller than the system size N , at each interface we have the phenomenology depicted previously in the infinite system case, that is a linear behaviour of the interface given by equations (46) and (47). One may also remark that some transverse magnetization is lost during the initial time $t \sim N$. Indeed, since the system at $t = 0$ is coupled with the bath chains, and since the total energy is conserved, a portion of the transverse energy term, $e^z = -h/2\sigma^z$, will relax into the coupling energy term, $e^{xx} = -1/2\sigma^x\sigma^x$, leading to a net loss of transverse magnetization. For times larger than the typical time $\tau = N$, no more transverse magnetization is lost and it behaves like a conserved quantity. From the numerical profiles we see the emergence of a flat profile in the middle of the system. The moving front between the flat finite transverse magnetization region and the zero magnetization region is linear with a slope $\sim 1/(3t)$ and has a constant width of size N . This leads, in the asymptotic regime $t \gg N$, to the scaling behaviour

$$m(n, t) \equiv \langle \sigma_n^z \rangle(t) = M_{tot} \frac{1}{2t} \Pi \left(\frac{n}{2t} \right), \quad (51)$$

where n is the site label measured from the middle of the system and where $M_{tot} = \sum_n m(n, t)$ is the total conserved transverse magnetization $M_{tot} = \sum_n m(n, t \gg N)$. Numerically we found evidences for $M_{tot} = N m_{bulk}^{z,sys}$, where $m_{bulk}^{z,sys} = 2/\pi$ is the $T = 0$ equilibrium system bulk transverse magnetization.

Finally in Figure 4, we show the results obtained in the droplet problem in the case where the bath part is at finite temperature. We have more or less the same scenario as in the infinite temperature bath case, that is a velocity one propagating front, which here is not linear, of constant size N . In the large time limit, $t \gg N$, numerically we obtain the scaling behaviour

$$m(n, t) = m_b^{z,bath} + \frac{N}{2t} \Pi \left(\frac{n}{2t} \right) k \left(\frac{n}{t} \right) \quad (52)$$

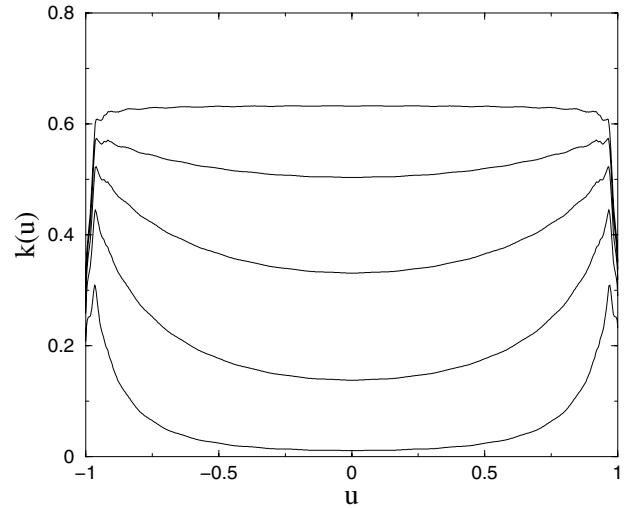


Fig. 5. Scaling function $k(u)$ obtained for $T_b = \infty$, $T_b = 5$, $T_b = 2$, $T_b = 1$, $T_b = 1/2$, from top to bottom.

with a temperature dependent function $k(u)$. Again, the total transverse magnetization is conserved in the asymptotic limit $t \gg N$ for the same reasons as discussed previously. If one defines the excess total magnetization as $\Delta M_{tot} = \sum_n [m(n, t \gg N) - m_b^{z,bath}]$, one obtains by integration of the profile (52) the relation $\Delta M_{tot} = \frac{N}{2} \int_{-1}^1 k(u) du$. In order to recover the scaling form (51), one has, in the limit $T_b \rightarrow \infty$, $k(u) \rightarrow 1$. In Figure 5 we show the scaling function $k(u)$ obtained for different temperatures of the bath. We see clearly the flattening of the profile as the temperature increases.

6 Discussion

We have investigated the front propagation and scaling profiles of transverse magnetization inhomogeneities in the Ising quantum chain. The inhomogeneities were generated initially by local equilibration with several thermal baths at different temperatures. We have concentrated our study on two distinct temperature configurations: the first one being the interface problem, where half of the chain is at a given temperature, the other half at an other temperature. The second situation considered is one, referred as the droplet problem, where initially a small part of the infinite chain is at a different temperature from the remaining part. In both cases, in the asymptotic time regime the transverse magnetization exhibits a scaling form $t^{-1} M(n/t)$, where the scaling function M depends on the initial temperatures and on the transverse field value h . At the critical field value, $h = 1$, in the extreme case $T_1 = 0$ and $T_2 = \infty$, the transverse magnetization front is linear, while within the droplet a flat profile emerges. Asymptotically, in the droplet situation the scaling profile is given by a characteristic function $\Pi(u)$. Another interesting feature of such relaxation problem is the scaling behaviour of the front itself. In particular, we have shown that in the large field situation, $h > 1$, the front has a

staircase structure with scaling $t^{-1/3}g((\ell-t)/t^{1/3})$, meaning that the area of each step is conserved which implies a quantized relaxation of the magnetization, as already found in the XX-chain in reference [17].

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Appendix

Introducing the $2L$ component operator Γ such that

$$\Gamma^\dagger = (\Gamma_1^1, \Gamma_1^2, \Gamma_2^1, \Gamma_2^2, \dots, \Gamma_L^1, \Gamma_L^2) \quad (53)$$

the Hamiltonian (1) is put in the quadratic form

$$\mathcal{H} = \frac{1}{4} \Gamma^\dagger T \Gamma \quad (54)$$

where T is a $2L \times 2L$ matrix. The diagonalization of the Hamiltonian can then be performed by the introduction of the diagonal Clifford operators γ_q^1 and γ_q^2 , related to the lattice operators via $\Gamma_n^1 = \sum_q \phi_q(n) \gamma_q^1$ and $\Gamma_n^2 = \sum_q \psi_q(n) \gamma_q^2$ with ϕ and ψ defined through the eigenvalue equation

$$T V_q = \epsilon_q V_q \quad (55)$$

with $V_q(2n-1) = \phi_q(n)$ and $V_q(2n) = -i\psi_q(n)$. It leads to the form

$$\mathcal{H} = i \sum_q \frac{\epsilon_q}{2} \gamma_q^1 \gamma_q^2. \quad (56)$$

The time evolution of the diagonal operators is obtained simply by $\gamma_q(t) = U_q^\dagger(t) \gamma_q U_q(t)$ with

$$U_q(t) = \exp\left(\frac{\epsilon_q t}{2} \gamma_q^1 \gamma_q^2\right) = \cos \frac{\epsilon_q t}{2} + \gamma_q^1 \gamma_q^2 \sin \frac{\epsilon_q t}{2}. \quad (57)$$

Utilizing the fact that $\{\gamma_q^i, \gamma_{q'}^j\} = 2\delta_{ij}\delta_{qq'}$, we obtain

$$\gamma_q^i(t) = \sum_{j=1}^2 \langle \gamma_q^j | \gamma_q^i(t) \rangle \gamma_q^j \quad (58)$$

where we have defined the pseudo-scalar product as

$$\langle C | D \rangle = \frac{1}{2} \{C^\dagger, D\} \quad (59)$$

with $\{.,.\}$ the anticommutator. The time-dependent lattice Clifford generators, $\Gamma_n^i(t)$, can then be re-expressed in terms of the initial time operators Γ with the help of the inverse transforms $\gamma_q^1 = \sum_k \phi_q(k) \Gamma_k^1$ and $\gamma_q^2 = \sum_k \psi_q(k) \Gamma_k^2$. Finally, one obtains

$$\Gamma_n^j(t) = \sum_{k,i} \langle \Gamma_k^i | \Gamma_n^j(t) \rangle \Gamma_k^i \quad (60)$$

with components

$$\begin{aligned} \langle \Gamma_k^1 | \Gamma_n^1(t) \rangle &= \sum_q \phi_q(k) \phi_q(n) \cos \epsilon_q t \\ \langle \Gamma_k^1 | \Gamma_n^2(t) \rangle &= \langle \Gamma_n^2 | \Gamma_k^1(-t) \rangle = - \sum_q \phi_q(k) \psi_q(n) \sin \epsilon_q t \\ \langle \Gamma_k^2 | \Gamma_n^2(t) \rangle &= \sum_q \psi_q(k) \psi_q(n) \cos \epsilon_q t. \end{aligned} \quad (61)$$

One may notice that the time evolution of the $2L$ -components operator Γ is given by a $2L$ -rotation matrix R with elements

$$R_{i,j}(t) = \langle \Gamma_j | \Gamma_i(t) \rangle \quad (62)$$

where $\Gamma_{2i-1} \equiv \Gamma_i^1$ and $\Gamma_{2i} \equiv \Gamma_i^2$. The time evolution of Γ is then

$$\Gamma(t) = R(t) \Gamma. \quad (63)$$

The set $\{\Gamma_k^i\}$ forms an orthonormal basis of a $2L$ -dimensional linear vector space \mathcal{E} with inner product defined by $\langle . | . \rangle \equiv \frac{1}{2} \{.,.\}$. Hence, every vector $X \in \mathcal{E}$ has a unique expansion $X = \sum_{i,k} \langle \Gamma_k^i | X \rangle \Gamma_k^i$. The string expression $X_1 X_2 \dots X_n$, with $X_j \in \mathcal{E}$, is a direct product vector of the space $\mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \dots \otimes \mathcal{E}_n$ which decomposition is

$$X_1 X_2 \dots X_n = \sum_{i_1, k_1, \dots, i_n, k_n} \langle \Gamma_{k_1}^{i_1} | X_1 \rangle \dots \langle \Gamma_{k_n}^{i_n} | X_n \rangle \Gamma_{k_1}^{i_1} \dots \Gamma_{k_n}^{i_n}. \quad (64)$$

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